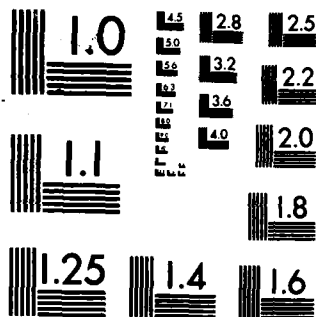MICROCOPY RESOLUTION TEST CHART
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Research Report CCS 547

AN EXTENSION OF THE FIBONACCI SERIES
APPLIED TO
A REPLENISHMENT PROBLEM

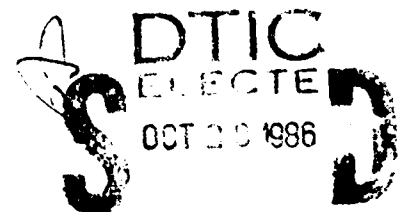by

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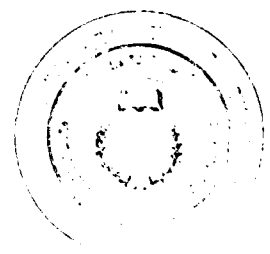
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AN EXTENSION OF THE FIBONACCI SERIES
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July, 1986

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ABSTRACT

This note introduces an extension of the Fibonacci series. The new series provides new interpretation for two problems of replenishment which appeared in the April 1986 issue of OR/MS Today which have been stated to be hitherto unsolved. The series offers closed form solutions to both problems.

KEYWORDS: Fibonacci Series, Combinatorics

Introduction

The following problem appeared in the April, 1986 issue of OR/MS Today.
It is a problem which immediately identifies itself as a combinatorial
problem.

> A backpacker wishes to cross the desert and can carry a five-day
> supply of food and water. He has <u>many</u> friends who are willing to
> help him and each of them can carry five days of food and water.
> Water and food cannot be stashed and each of his friends wants to
> return home. Each day out, each person consumes a full day of food
> and water. How many days of food and water are necessary to cross
> the desert if the journey requires n days?
> a. n = 7 - rather easy
> b. n = 8 - a moderate challenge
> c. n = 9 - a real challenge
> d. n = 10 - unsolved
> e. Solve the previous problems for the case when the
>   backpacker must return from the summit of the mountain.

Both problems, that of the desert crossing and the summit of the
mountain, lend themselves to interpretation via a series of numbers in-
troduced in this discussion. The series can, quite simply, be thought of
as an extension of the Fibonacci series. The use of this series allows
a closed form expression to the <u>general problem of n-day desert crossings
or n-day summit of the mountain trips</u>. The following discussion is sup-
ported by figures which illustrate the actual patterns of replenishment.

The backpacker's problem can be conceptualized as a problem of re-
plenishment. Since the packer (and his friends) can carry a maximum of
5 units of food and water, a journey of n > 5 days requires that the packer
be replenished by n-5 units of food and water. By adopting the (intui-
tive) reasoning that the packer must be replenished as quickly as possi-
ble, it is necessary to determine how many days of food and water are
required for the packer to be replenished by 1 unit at the end of day 1,

day 2, ..., day n-5. Consider Figure 1, which pictorially represents the replenishment process for a 7-day journey. Two replenishers are required to send the backpacker on his trip, the first carrying a load of 3 units and the second carrying a load of 5 units. The first replenisher uses one unit on the first day, a second unit goes to replenish the backpacker and the third unit is used for the return of this replenisher. Similarly, the second replenisher uses two units for his forward journey, two units for his return journey and one unit goes to replenish the backpacker on day 2.

The Replenishment Series T(k).

Defining

$T(k)$ = Number of units of food and water required to replenish the packer by 1 unit of food and water on day k,

we have,

k	T(k)	$T(k)=a_k T(2)+b_k T(1)$
1	3	$0T(2) + 1T(1)$
2	5	$1T(2) + 0T(1)$
3	$5 + 2(T(1))$	$1T(2) + 2T(1)$
4	$5 + 2(T(1)+T(2))$	$3T(2) + 2T(1)$
5	$5 + 2(T(1)+T(2)+T(3))$	$5T(2) + 6T(1)$
.	.	.
.	.	.
.	.	.

Thus,

$$T(k) = 5 + 2 \sum_{i=1}^{k-2} T(i), \qquad k > 2$$

or,

$$T(k) = T(k-1) + 2T(k-2), \quad k \geq 2$$

where,

$$T(0) = 1$$

The recursion defines a series much akin to the Fibonacci series which, of course, is defined by the recursion $F(k) = F(k-1) + F(k-2)$. $T(k)$ may be further expressed in terms of $T(2)$ and $T(1)$ as follows,

$$T(k) = a_k T(2) + b_k T(1),$$

where,

$$a_1 = 0, \ a_2 = 1$$
$$b_1 = 1, \ b_2 = 0$$

and,

$$a_k = a_{k-1} + b_{k-1}, \quad k \geq 2,$$
$$b_k = 2a_{k-1}, \quad k \geq 2.$$

Further, note that $|a_k - b_k| = 1$.

Solution to the Desert Crossing Problem

The backpacker must be replenished on days $1, 2, \ldots, n-5$, $(n>5)$ in order to make an n-day journey across the desert. Letting

$S(n)$ = Number of units of food and water to required to
make an n-day journey,

we have,

$$S(n) = 5 + \sum_{i=1}^{n-5} T(i), \qquad n \geq 5.$$

The replenishment process is represented pictorially in Figures 1-4 for $n = 7,8,9,10$ respectively. To illustrate, consider the 7-day journey represented in Figure 7. The packer begins with a full load, is replenished at the end of day 1 by a buddy who began with a load of 3 units of

3

food and water. At the end of day 2, he is replenished by another buddy who began the journey with 5 units of food and water and thus is able to complete the 7-day journey. Note that in Figures 2-4, replenishers for days 3 onwards require replenishing themselves in order to make the trip back. To illustrate, for the 8-day journey in Figure 2, the day 3 replenisher must be replenished by 1 unit at the end of his fifth day, thus requiring this replenisher to have left base camp on day 5 allowing both of them to return at the end of day 6.

Solution to the Summit of the Mountain Problem

When the backpacker must return, we again make use of the replenishment function $T(k)$. This function represents a returning backpacker with the provision of returning with one excess day of food and water. This excess unit can supply the backpacker with an additional unit of food and water thus eliminating the need for one units worth of replenishment. When the backpacker is finally loaded with five units, he can go forward for three days and return for two days without further replenishment or go forward for two days and return for three days without replenishment. Thus a returning backpacker is a day-n replenisher who does not need further replenishing himself on the n-2 leg of his return journey. Defining,

R(n) = Number of units of food and water to required to make an n-day return journey,

we have,

$$R(n) = T(n) - T(n-2) \qquad n \geq 2$$

The replenishment process is represented pictorially in Figures 5 and 6 for n = 4 and 5 respectively.
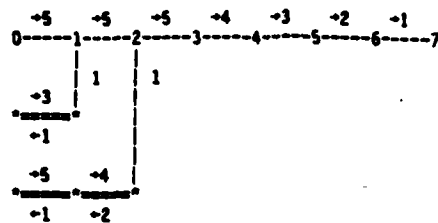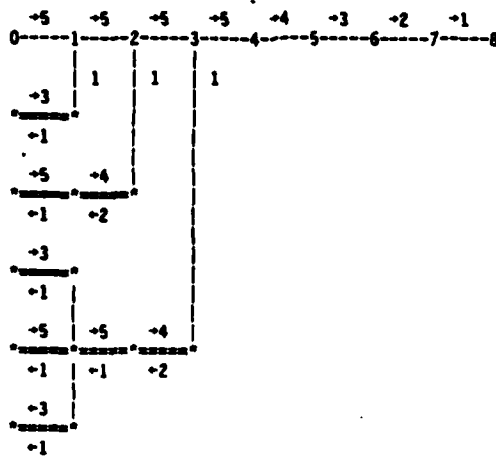
Figure 1. Replenishment problem for 7-day journey
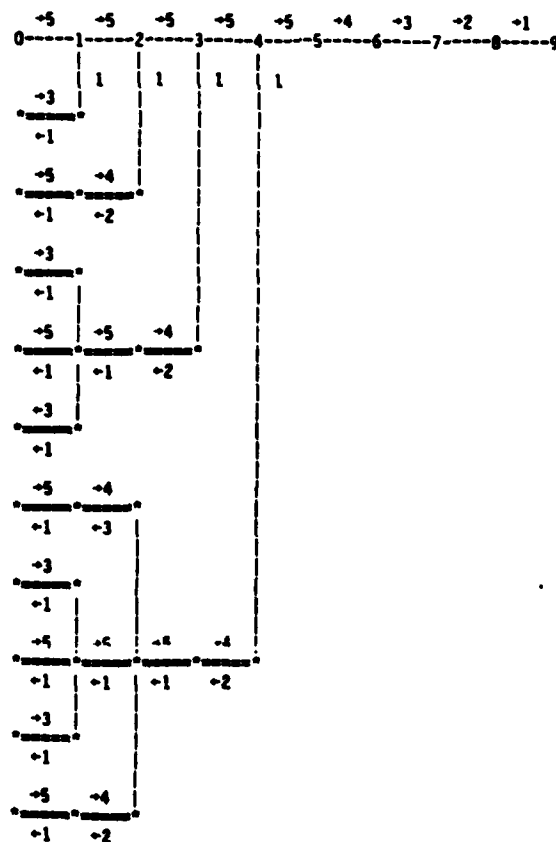

Figure 2. Replenishment problem for 8-day journey

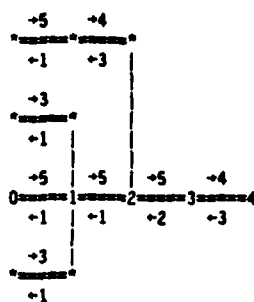Figure 3. Replenishment problem for 9-day journey

Figure 4. Replenishment problem for 10-day journey

```
        →5      →4
     *═══════*═══════*
      ←1      ←3     │
                     │
        →3           │
     *═══════*       │
      ←1     │       │
            │        │
        →5  │   →5   │   →5      →4
     0═══════1═══════2═══════3═══════4
      ←1    │  ←1     ←2      ←3
           │
        →3 │
     *═══════*
      ←1
```

Figure 5. Replenishment problem for 4-day return journey

```
        →3
     *═══════*
      ←1     │
            │
        →5  │   →5      →4
     *═══════*═══════*═══════*
      ←1    │  ←1     ←2     │
           │               │
        →3 │               │
     *═══════*             │
      ←1                   │
                          │
        →5      →4         │
     *═══════*═══════*     │
      ←1      ←2    │      │
                  │       │
        →3        │       │
     *═══════*    │       │
      ←1         │        │
                │         │
        →5      │   →5    │   →5    │   →5      →4
     0═══════1═══════2═══════3═══════4═══════5
      ←1     │  ←1     ←1   │  ←2      ←3
            │              │
        →3  │              │
     *═══════*             │
      ←1                   │
                          │
        →5      →4         │
     *═══════*═══════*     │
      ←1      ←2
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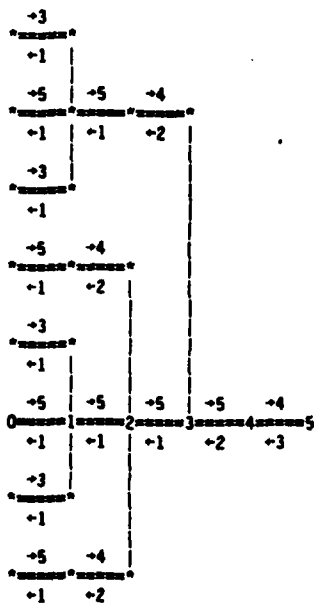
Figure 6. Replenishment problem for 5-day return journey.

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